

Bifurcation diagrams and critical subsystems of the Kowalevski gyrostat in two constant fields

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Abstract

The Kowalevski gyrostat in two constant fields is known as the unique example of an integrable Hamiltonian system with three degrees of freedom not reducible to a family of systems in fewer dimensions and still having the clear mechanical interpretation. The practical explicit integration of this system can hardly be obtained by the existing techniques. Then the challenging problem becomes to fulfil the qualitative investigation based on the study of the Liouville foliation of the phase space. As the first approach to topological analysis of this system we find the stratified critical set of the momentum map; this set consists of the trajectories with number of frequencies less than three. We obtain the equations of the bifurcation diagram in three-dimensional space. These equations have the form convenient for the classification of the bifurcation sets induced on 5-dimensional iso-energetic levels.

MSC: 70E17, 70G40, 70H06

PACS: 45.20.Jj, 45.40.Cc

Key Words: Kowalevski gyrostat, two constant fields, critical set, bifurcation diagram

1 Introduction

The famous integrable case of S. Kowalevski of the motion of a heavy rigid body about a fixed point [1] has received several generalizations. Some of them suppose restrictions to submanifolds in the phase space (partial cases), others are far from mechanics involving potential functions on the configuration space $SO(3)$ with singularities. The most essential generalization having the clear mechanical sense was found by A.G. Reyman and M.A. Semenov-Tian-Shansky in the work [2]. The authors introduce the dynamical system on the dual space of the Lie algebra $e(p, q)$ of the Lie group defined as the semi-direct product of $SO(p)$ and q copies of \mathbb{R}^p . Such systems are known as the Euler equations on Lie (co)algebras [3]. The case $p = 3, q = 2$ corresponds to the Euler–Poisson equations of the motion of a gyrostat in two constant fields.

For a rigid body without gyrostatic momentum, the model of two constant fields was introduced by O.I. Bogoyavlensky [3]. The physical object can be either a heavy electrically charged rigid body rotating in gravitational and constant electric fields, or a heavy magnet rotating in gravitational and constant magnetic fields. The corresponding equations are Hamiltonian on the orbit of coadjoint action on $e(3, 2)^*$ of the Lie group defined as the semi-direct product $SO(3) \times (\mathbb{R}^3 \otimes \mathbb{R}^3)$. The typical orbit is diffeomorphic to $TSO(3) \cong \mathbb{R}^3 \times SO(3)$. Therefore, the gyrostat in two constant fields is the Hamiltonian system with three degrees of freedom. Bogoyavlensky [3] suggested the conditions of the Kowalevski type and found the analogue of the Kowalevski integral K for the top in two constant fields. H. Yehia [4] generalized this integral for the Kowalevski gyrostat in two constant fields. Almost simultaneously with Yehia, I.V. Komarov [5] and L.N. Gavrilov [6] proved the Liouville integrability of the Kowalevski gyrostat in the gravity field. But for two constant fields the Kowalevski gyrostat was not considered integrable due to the fact that the existence of the second field destroys the axial symmetry of the potential and, consequently, the corresponding cyclic integral. Finally, Reyman and

Semenov-Tian-Shansky [2] found the Lax representation with a spectral parameter for the family of Euler equations on $e(p, q)^*$. For $e(3, 2)$ this representation immediately gave rise to the new integral for the Kowalevski gyrostat in two constant fields. For the classical Kowalevski top this integral turns into the square of the cyclic integral.

The Kowalevski gyrostat in two constant fields does not have any explicit symmetry groups and, therefore, is not reducible, in a standard way, to a family of systems with two degrees of freedom. Phase topology of such systems has not been studied yet. The theory of n -dimensional integrable systems started in [7] is not illustrated by an application to any real irreducible physical or geometrical problem with $n > 2$.

In the paper [8], the authors give a detailed exposition of the results of [2] as well as a study of the algebraic geometry of the Lax pair for the generalized Kowalevski system. They announce the possibility of its integration by the finite-band techniques and fulfil such integration for the classical top. For two constant fields the integration of the Kowalevski top is not given up-to-date. The problem of the Kowalevski *gyrostat* motion in two constant fields is not studied at all. The technical difficulties here are extremely high. It is not likely that, in the general regular case, the analytical solutions can be obtained having the form useful for the qualitative topological analysis or the computer simulation. However, there is a good experience of studying the critical subsystems, i.e., the systems with two degrees of freedom induced on 4-dimensional invariant submanifolds of the phase space. For the Kowalevski top in two constant fields we have now the complete description of all singularities of the momentum map [9], [10], [11], [12], [13], [14] and the classification of the bifurcation diagrams for the restriction of this map to 5-dimensional iso-energetic surfaces [15], [16], [17]. This result is a necessary and highly complicated part of the study of Liouville foliation of the integrable system and shows the actual need in the generalization of the Liouville invariants theory [18] for the dimensions greater than two.

The present paper contains similar results for the Kowalevski gyrostat in two constant fields. The 6-dimensional phase space is stratified by the rank of the momentum map. We find the equations of invariant submanifolds on which the induced systems are Hamiltonian with less than three degrees of freedom (critical manifolds of rank 0, 1, or 2). We straightforwardly prove that the image of these critical manifolds (the bifurcation diagram) lies in the discriminant set of the algebraic curve of the Lax representation given in [8]. Moreover, the spectral parameter on the Lax curve is explicitly expressed in terms of the constant s of the additional partial integral arising on the critical submanifolds. It then follows that the equations of the surfaces containing the bifurcation diagram are written in the parametric form such that the parameters are the energy constant h and the constant s of the partial integral. Fixing the value of h we come to explicit equations of the bifurcation diagrams induced on iso-energetic levels. The problem of classification of these diagrams seems quite complicated due to the existence of several physical parameters. Nevertheless, it is certainly solvable with the help of contemporary computer programs of analytical calculations.

First we show that the number of physical parameters for the gyrostat in two constant fields can be reduced by a simple procedure, which may be called the orthogonalization of the fields. More precisely, for the problems of gyrostat motion there exists a group of diffeomorphisms of the phase spaces (mentioned above orbits of the coadjoint action) that is an equivalence group for the corresponding dynamical systems. It appears that each equivalence class contains a problem with an orthonormal pair of radius vectors of the centers of forces application and with an orthogonal pair of the intensity vectors. Such force field is characterized by only one essential parameter—the ratio of the modules of the intensity vectors. For a dynamically symmetric gyrostat having the centers of forces application in the equatorial plane, the orthogonalization procedure along with the appropriate choice of the measure units leave, in addition to the forces ratio, only two physical parameters of the body itself, namely, the ratio of the equatorial and axial inertia moments and the non-zero axial component of the gyrostatic momentum. In the generalized Kowalevski case the first of them equals 2. Thus, the whole problem has, in fact, two essential parameters. In particular, each of the critical four-dimensional submanifolds found below provides a two-parametric family of completely integrable Hamiltonian systems with two degrees of freedom.

2 Gyrostat equations and parametrical reduction

Consider a rigid body \mathcal{B} rotating around a fixed point O . Choose a trihedral at O moving along with the body and refer to it all vector and tensor objects. Denote by $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ the canonical unit basis in \mathbb{R}^3 ; then the moving trihedral itself is represented as $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Let $\boldsymbol{\omega}$ be the vector of the angular velocity of \mathcal{B} . Suppose that \mathcal{B} is bearing an axially symmetric rigid rotor \mathcal{B}' rotating freely around its symmetry axis fixed in \mathcal{B} . Such system of two bodies is the simplest model of a gyrostat. The notion of a gyrostat was introduced by N.E. Zhukovsky [19] for a body having cavities totally filled with homogeneous fluid. Both models have the common feature usually taken as the definition of a gyrostat: the total angular momentum of such system is $\mathbf{M} = \mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}$, where the inertia tensor \mathbf{I} and the vector $\boldsymbol{\lambda}$ (called the gyrostatic momentum) are constant with respect to the moving trihedral. Using the term "gyrostat" we always suppose $\boldsymbol{\lambda} \neq 0$. In the case $\boldsymbol{\lambda} = 0$ we use the terms "rigid body" or "top" instead. The top is usually supposed to have a dynamical symmetry axis.

Let \mathbf{M}_F denote the moment of external forces with respect to O (the rotating moment). Constant field is a force field inducing the rotating moment of the form $\mathbf{r} \times \boldsymbol{\alpha}$ with constant vector \mathbf{r} and with $\boldsymbol{\alpha}$ corresponding to some physical vector fixed in inertial space; \mathbf{r} points from O to the center of application of the field, $\boldsymbol{\alpha}$ is the field intensity.

For *two constant fields* the rotating moment is $\mathbf{M}_F = \mathbf{r}_1 \times \boldsymbol{\alpha} + \mathbf{r}_2 \times \boldsymbol{\beta}$ with $\mathbf{r}_1, \mathbf{r}_2$ constant in the body and $\boldsymbol{\alpha}, \boldsymbol{\beta}$ corresponding to the vectors fixed in inertial space. Obviously, \mathbf{M}_F can be represented as the moment of *one constant field* if either $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ or $\boldsymbol{\alpha} \times \boldsymbol{\beta} = 0$. Suppose that

$$\mathbf{r}_1 \times \mathbf{r}_2 \neq 0, \quad \boldsymbol{\alpha} \times \boldsymbol{\beta} \neq 0. \quad (2.1)$$

Two constant fields satisfying (2.1) are said to be *independent*.

The equations defining the respective evolution of $\mathbf{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}$ in two constant fields are

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \boldsymbol{\omega} + \mathbf{r}_1 \times \boldsymbol{\alpha} + \mathbf{r}_2 \times \boldsymbol{\beta}, \quad \frac{d\boldsymbol{\alpha}}{dt} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \frac{d\boldsymbol{\beta}}{dt} = \boldsymbol{\beta} \times \boldsymbol{\omega}. \quad (2.2)$$

These equations are Euler equations in the space $\mathbb{R}^9(\mathbf{M}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ considered as the dual space to the semi-direct sum $so(3) + (\mathbb{R}^3 \otimes \mathbb{R}^3)$. The Lie–Poisson bracket applied to the coordinate functions yields

$$\begin{aligned} \{M_i, M_j\} &= \varepsilon_{ijk} M_k, & \{M_i, \alpha_j\} &= \varepsilon_{ijk} \alpha_k, & \{M_i, \beta_j\} &= \varepsilon_{ijk} \beta_k, \\ \{\alpha_i, \alpha_j\} &= 0, & \{\alpha_i, \beta_j\} &= 0, & \{\beta_i, \beta_j\} &= 0. \end{aligned} \quad (2.3)$$

Such bracket is non-degenerate on each orbit of the coadjoint action. The orbits are defined by the geometric integrals (common level of the Casimir functions)

$$\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = c_{11}, \quad \boldsymbol{\beta} \cdot \boldsymbol{\beta} = c_{22}, \quad \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = c_{12}.$$

If $c_{11} > 0, c_{22} > 0, c_{12}^2 < c_{11}c_{22}$, then the orbit in \mathbb{R}^9 is diffeomorphic to $\mathbb{R}^3 \times SO(3)$, and the induced Hamiltonian system has three degrees of freedom (see [3], [8] for the details). From physical point of view the constants c_{11}, c_{22}, c_{12} characterize the force fields intensities. Along with the coordinates of $\mathbf{r}_1, \mathbf{r}_2$ in the moving frame, we have 9 parameters of the interaction of the body with the external forces. We now show how to reduce this number.

Introduce some notation.

Let $L(n, k)$ be the space of $n \times k$ -matrices. Put $L(k) = L(k, k)$.

Identify $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ with $L(3, 2)$ by the isomorphism j that joins two columns

$$A = j(\mathbf{a}_1, \mathbf{a}_2) = \|\mathbf{a}_1 \ \mathbf{a}_2\| \in L(3, 2), \quad \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3.$$

For the inverse map, we write

$$j^{-1}(A) = (\mathbf{c}_1(A), \mathbf{c}_2(A)) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad A \in L(3, 2).$$

If $A, B \in L(3, 2)$, $\mathbf{a} \in \mathbb{R}^3$, by definition, put

$$\begin{aligned} A \times B &= \sum_{i=1}^2 \mathbf{c}_i(A) \times \mathbf{c}_i(B) \in \mathbb{R}^3; \\ \mathbf{a} \times A &= j(\mathbf{a} \times \mathbf{c}_1(A), \mathbf{a} \times \mathbf{c}_2(A)) \in L(3, 2). \end{aligned} \quad (2.4)$$

Lemma 1. *Let $\Lambda \in SO(3)$, $D \in GL(2, \mathbb{R})$, $\mathbf{a} \in \mathbb{R}^3$, $A, B \in L(3, 2)$. Then*

$$\begin{aligned} \Lambda(A \times B) &= (\Lambda A) \times (\Lambda B); \quad (\Lambda D^{-1}) \times (B D^T) = A \times B; \\ \Lambda(\mathbf{a} \times A) &= (\Lambda \mathbf{a}) \times (\Lambda A); \quad \mathbf{a} \times (\Lambda D) = (\mathbf{a} \times A) D. \end{aligned}$$

The proof is by direct calculation.

In notation (2.4) we write Eqs. (2.2) in the form

$$\mathbf{I} \frac{d\boldsymbol{\omega}}{dt} = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \times \boldsymbol{\omega} + A \times U, \quad \frac{dU}{dt} = -\boldsymbol{\omega} \times U. \quad (2.5)$$

Here $A = j(\mathbf{r}_1, \mathbf{r}_2)$ is a constant matrix, $U = j(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The phase space of (2.5) is $\{(\boldsymbol{\omega}, U)\} = \mathbb{R}^3 \times L(3, 2)$.

In fact, U in (2.5) is restricted by the geometric integrals; i.e., for some constant symmetric matrix $C \in L(2)$

$$U^T U = C. \quad (2.6)$$

Let \mathcal{O} be the set defined by Eq. (2.6) in $L(3, 2)$. In order to emphasize the C -dependence, we write $\mathcal{O} = \mathcal{O}(C)$.

Let $\mathfrak{P} = (\mathbf{I}, \boldsymbol{\lambda}, A, C)$ denote the complete set of constant parameters of the problem. Denote by $X_{\mathfrak{P}}$ the vector field on $\mathbb{R}^3 \times \mathcal{O}(C)$ induced by (2.5). Given the set \mathfrak{P} , the problem of motion of the gyrostat in two constant fields described by the dynamical system $X_{\mathfrak{P}}$ will be called, for short, the *DG-problem*.

Associate to $\Lambda \in SO(3)$, $D \in GL(2, \mathbb{R})$ the linear automorphisms $\Psi(\Lambda, D)$ and $\psi(\Lambda, D)$ of $\mathbb{R}^3 \times L(3, 2)$ and $L(3) \times \mathbb{R}^3 \times L(3, 2) \times L(2)$

$$\begin{aligned} \Psi(\Lambda, D)(\boldsymbol{\omega}, U) &= (\Lambda \boldsymbol{\omega}, \Lambda U D^T), \\ \psi(\Lambda, D)(\mathbf{I}, \boldsymbol{\lambda}, A, C) &= (\Lambda \mathbf{I} \Lambda^T, \Lambda \boldsymbol{\lambda}, \Lambda A D^{-1}, D C D^T). \end{aligned} \quad (2.7)$$

Eqs. (2.6) and (2.7) imply $\Psi(\Lambda, D)(\mathbb{R}^3 \times \mathcal{O}(C)) = \mathbb{R}^3 \times \mathcal{O}(D C D^T)$. Using Lemma 1 we obtain the following statement.

Lemma 2. *For each $(\Lambda, D) \in SO(3) \times GL(2, \mathbb{R})$, we have*

$$\Psi(\Lambda, D)_*(X_{\mathfrak{P}}(v)) = X_{\psi(\Lambda, D)(\mathfrak{P})}(\Psi(\Lambda, D)(v)), \quad v \in \mathbb{R}^3 \times \mathcal{O}(C).$$

Thus, any two DG-problems determined by the sets of parameters \mathfrak{P} and $\psi(\Lambda, D)(\mathfrak{P})$ are completely equivalent.

Let us call a DG-problem *canonical* if the centers of application of forces lie on the first two axes of the moving trihedral at unit distance from O and the intensities of the forces are orthogonal to each other.

Theorem 1. *For each DG-problem with independent forces there exists an equivalent canonical problem. Moreover, in both equivalent problems the centers of application of forces belong to the same plane in the body containing the fixed point.*

Proof. Let the DG-problem with the set of parameters $\mathfrak{P} = (\mathbf{I}, \boldsymbol{\lambda}, A, C)$ satisfy (2.1). This means that the symmetric matrices $A_* = (A^T A)^{-1}$ and C are positively definite. According to the well-known fact from linear algebra, A_* and C can be reduced, respectively, to the identity matrix and to a diagonal matrix via the same conjugation operator

$$D A_* D^T = E, \quad D C D^T = \text{diag}\{a^2, b^2\}, \quad D \in GL(2, \mathbb{R}), \quad a, b \in \mathbb{R}_+.$$

Then $\mathbf{c}_1(AD^{-1})$ and $\mathbf{c}_2(AD^{-1})$ form an orthonormal pair in \mathbb{R}^3 . There exists $\Lambda \in SO(3)$ such that $\Lambda \mathbf{c}_i(AD^{-1}) = \mathbf{e}_i$ ($i = 1, 2$). The first statement is obtained by applying Lemma 2 with the previously chosen Λ, D to the initial vector field $X_{\mathfrak{p}}$.

To finish the proof, notice that the transformation $A \mapsto AD^{-1}$ preserves the span of $\mathbf{c}_1(A)$, $\mathbf{c}_2(A)$. The matrix Λ in (2.7) stands for the change of the moving trihedral. Therefore, if $\mathbf{a} \in \mathbb{R}^3$ represents some physical vector in the initial problem, then $\Lambda \mathbf{a}$ is the same vector with respect to the body in the equivalent problem. \square

Remark 1. The fact that any DG-problem can be reduced to the problem with *one* of the pairs $\mathbf{r}_1, \mathbf{r}_2$ or $\boldsymbol{\alpha}, \boldsymbol{\beta}$ orthonormal is obvious. Simultaneous orthogonalization of *both* pairs was first established in [11] for a rigid body and crucially simplifies all calculations.

It follows from Theorem 1 that, without loss of generality, for independent forces we may suppose

$$\mathbf{r}_1 = \mathbf{e}_1, \quad \mathbf{r}_2 = \mathbf{e}_2, \quad (2.8)$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = a^2, \quad \boldsymbol{\beta} \cdot \boldsymbol{\beta} = b^2, \quad \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0. \quad (2.9)$$

Change, if necessary, the order of $\mathbf{e}_1, \mathbf{e}_2$ (with simultaneous change of the direction of \mathbf{e}_3) to obtain $a \geq b > 0$.

Consider a dynamically symmetric top in two constant fields with the centers of application of forces in the equatorial plane of its inertia ellipsoid. Choose a moving trihedral such that $O\mathbf{e}_3$ is the symmetry axis. Then the inertia tensor \mathbf{I} becomes diagonal. Let $a = b$. For any $\Theta \in SO(2)$ denote by $\hat{\Theta} \in SO(3)$ the corresponding rotation of \mathbb{R}^3 about $O\mathbf{e}_3$. Take in (2.7) $\Lambda = \hat{\Theta}$, $D = \Theta$. Under the conditions (2.8), (2.9), $\psi = \text{Id}$ and Ψ becomes the symmetry group. The system (2.5) has the cyclic integral $\mathbf{I}\boldsymbol{\omega} \cdot (a^2\mathbf{e}_3 - \boldsymbol{\alpha} \times \boldsymbol{\beta})$. Therefore it is possible to reduce such a DG-problem to a family of systems with two degrees of freedom. For the analogue of the Kowalevski case this system becomes integrable [4].

Let us call a DG-problem *irreducible* if, in its canonical representation,

$$a > b > 0. \quad (2.10)$$

The following statements are needed in the future; they also reveal some features of a wide class of DG-problems.

Lemma 3. *In an irreducible DG-problem, the body has exactly four equilibria.*

Proof. The set of singular points of (2.5) is defined by $\boldsymbol{\omega} = 0$, $A \times U = 0$. For the equivalent canonical problem with (2.8) we have

$$\mathbf{e}_1 \times \boldsymbol{\alpha} + \mathbf{e}_2 \times \boldsymbol{\beta} = 0.$$

Then the four vectors $\mathbf{e}_1, \boldsymbol{\alpha}, \mathbf{e}_2, \boldsymbol{\beta}$ are parallel to the same plane and $|\mathbf{e}_1 \times \boldsymbol{\alpha}| = |\mathbf{e}_2 \times \boldsymbol{\beta}|$. Given (2.10), this equality yields

$$\boldsymbol{\alpha} = \pm a\mathbf{e}_1, \quad \boldsymbol{\beta} = \pm b\mathbf{e}_2. \quad (2.11)$$

Thus, in the canonical irreducible system, an equilibrium takes place only if the radius vectors of the centers of application are parallel to the corresponding fields intensities. \square

Note that the existence of the gyrostatic momentum does not change the equilibria. Therefore, the result here is the same as in the case of a rigid body in two constant fields [15].

Lemma 4. *Let an irreducible DG-problem in its canonical form have the diagonal inertia tensor $\mathbf{I} = \text{diag}\{I_1, I_2, I_3\}$ and $\boldsymbol{\lambda} = 0$. Then the body has the following families of periodic motions of*

pendulum type

$$P_1 : \begin{cases} \boldsymbol{\omega} = \dot{\varphi} \mathbf{e}_1, & \boldsymbol{\alpha} \equiv \pm a \mathbf{e}_1, & \boldsymbol{\beta} = b(\mathbf{e}_2 \cos \varphi - \mathbf{e}_3 \sin \varphi), \\ & I_1 \dot{\varphi}'' = -b \sin \varphi; \end{cases} \quad (2.12)$$

$$P_2 : \begin{cases} \boldsymbol{\omega} = \dot{\varphi} \mathbf{e}_2, & \boldsymbol{\beta} \equiv \pm b \mathbf{e}_2, & \boldsymbol{\alpha} = a(\mathbf{e}_1 \cos \varphi + \mathbf{e}_3 \sin \varphi), \\ & I_2 \dot{\varphi}'' = -a \sin \varphi; \end{cases} \quad (2.13)$$

$$P_3 : \begin{cases} \boldsymbol{\omega} = \dot{\varphi} \mathbf{e}_3, & \boldsymbol{\alpha} \times \boldsymbol{\beta} \equiv \pm ab \mathbf{e}_3, \\ \boldsymbol{\alpha} = a(\mathbf{e}_1 \cos \varphi - \mathbf{e}_2 \sin \varphi), & \boldsymbol{\beta} = \pm b(\mathbf{e}_1 \sin \varphi + \mathbf{e}_2 \cos \varphi), \\ & I_3 \dot{\varphi}'' = -(a \pm b) \sin \varphi. \end{cases} \quad (2.14)$$

If $\boldsymbol{\lambda} \neq 0$ but $\boldsymbol{\lambda} = \lambda \mathbf{e}_i$ for some $i = 1, 2, 3$, then the only family remained is P_i with the corresponding index.

The proof is obvious. The families (2.12)–(2.14) were first found in [11] (the case $\boldsymbol{\lambda} = 0$). Note that for two constant fields these families are the only motions with a fixed direction of the angular velocity. In particular, the body in two independent constant fields does not have any uniform rotations.

3 Critical set of the Kowalevski gyrostat

Suppose that the irreducible DG-problem has the diagonal inertia tensor with the principal moments of inertia satisfying the ratio 2:2:1, the gyrostatic momentum is directed along the dynamical symmetry axis $\boldsymbol{\lambda} = \lambda \mathbf{e}_3$ and the centers of the fields application lie in the equatorial plane $\mathbf{r}_1 \perp \mathbf{e}_3, \mathbf{r}_2 \perp \mathbf{e}_3$. These are the conditions of the integrable case [2] of the Kowalevski gyrostat in two constant fields. The orthogonalization procedure in this case does not change the \mathbf{e}_3 -axis and we obtain (2.8), (2.9). Choosing the appropriate units of measurement, represent Eqs. (2.5) in the form

$$\begin{aligned} 2\dot{\omega}_1 &= \omega_2(\omega_3 - \lambda) + \beta_3, & 2\dot{\omega}_2 &= -\omega_1(\omega_3 - \lambda) - \alpha_3, & \dot{\omega}_3 &= \alpha_2 - \beta_1, \\ \dot{\alpha}_1 &= \alpha_2\omega_3 - \alpha_3\omega_2, & \dot{\beta}_1 &= \beta_2\omega_3 - \beta_3\omega_2, \\ \dot{\alpha}_2 &= \alpha_3\omega_1 - \alpha_1\omega_3, & \dot{\beta}_2 &= \beta_3\omega_1 - \beta_1\omega_3, \\ \dot{\alpha}_3 &= \alpha_1\omega_2 - \alpha_2\omega_1, & \dot{\beta}_3 &= \beta_1\omega_2 - \beta_2\omega_1. \end{aligned} \quad (3.15)$$

The phase space is $P^6 = \mathbb{R}^3 \times \mathcal{O}$, where $\mathcal{O} \subset \mathbb{R}^3 \times \mathbb{R}^3$ is defined by (2.9); \mathcal{O} is diffeomorphic to $SO(3)$.

The complete set of the first integrals in involution on P^6 includes the energy integral H , generalized Kowalevski integral K [3], [4], and the integral G found in [2]. After the parametrical reduction, these integrals are

$$\begin{aligned} H &= \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - \alpha_1 - \beta_2, \\ K &= (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2 + \\ &\quad + 2\lambda[(\omega_3 - \lambda)(\omega_1^2 + \omega_2^2) + 2\omega_1\alpha_3 + 2\omega_2\beta_3], \\ G &= \frac{1}{4}(M_\alpha^2 + M_\beta^2) + \frac{1}{2}(\omega_3 - \lambda)M_\gamma - b^2\alpha_1 - a^2\beta_2. \end{aligned}$$

Here $M_\alpha = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \cdot \boldsymbol{\alpha}$, $M_\beta = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \cdot \boldsymbol{\beta}$, $M_\gamma = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta})$.

Introduce the momentum map

$$J = G \times K \times H : P^6 \rightarrow \mathbb{R}^3 \quad (3.16)$$

and denote by $\mathfrak{C} \subset P^6$ the set of critical points of J . By definition, the bifurcation diagram of J is the set $\Sigma \subset \mathbb{R}^3$ over which J fails to be locally trivial; Σ defines the cases when the integral manifolds

$$J_c = J^{-1}(c), \quad c = (g, k, h) \in \mathbb{R}^3$$

change its topological (and smooth) type. To find \mathfrak{C} and Σ is the necessary part of the global topological analysis of the problem.

It follows from Liouville–Arnold theorem that for $c \notin \Sigma$ the manifold J_c , if not empty, is the union of three-dimensional tori. The considered Hamiltonian system on P^6 is non-degenerate at least for small enough values of b . Therefore the trajectories on such tori are almost everywhere quasi-periodic with three independent frequencies. The critical set \mathfrak{C} is preserved by the phase flow and consists of the trajectories having less than three frequencies. We call these trajectories *the critical motions*. The set \mathfrak{C} is stratified by the rank of J . Let $\mathfrak{C}_j = \{\zeta \in \mathfrak{C} : \text{rank } J(\zeta) = j\}$ ($j = 0, 1, 2$). It is natural to expect that \mathfrak{C}_j consists of the Liouville tori of dimension j and the image $J(\mathfrak{C}_j)$, as a subset of Σ , is a smooth surface Σ_j of dimension j . More precisely, for each $j \leq 2$ we have to take

$$\Sigma_j = J(\mathfrak{C}_j) \setminus \bigcup_{i=0}^{j-1} J(\mathfrak{C}_i).$$

Then, as a whole, we may consider Σ as a two-dimensional cell complex, Σ_j as its j -skeleton. For $j = 1, 2$ we will have $\partial\Sigma_j \subset \Sigma_{j-1}$.

For $c \in \Sigma_2$ the set $J_c \cap \mathfrak{C}$ consists of two-dimensional tori. Take the union of such tori over the values c from some open subset in Σ_2 . The dynamical system induced on this union will be Hamiltonian with two degrees of freedom. Vice versa, let M be a submanifold in P^6 , $\dim M = 4$, and suppose that the induced system on M is Hamiltonian. Then obviously $M \subset \mathfrak{C}$. This speculation gives a useful tool to find out whether a common level of functions consists of critical points of J .

Lemma 5. *Consider a system of equations*

$$f_1 = 0, \dots, f_{2k} = 0 \quad (3.17)$$

on a domain W open in the phase space P^{2n} of the integrable Hamiltonian system X . Let $M \subset W$ be the set defined by (3.17). Suppose

- (i) f_1, \dots, f_{2k} are smooth functions independent on M ;
- (ii) $Xf_1 = 0, \dots, Xf_{2k} = 0$ on M ;
- (iii) *the matrix of the Poisson brackets $\|\{f_i, f_j\}\|$ is non-degenerate almost everywhere on M .*

Then M consists of critical points of the momentum map.

Proof. Conditions (i), (ii) imply that M is a smooth $(2n - 2k)$ -dimensional manifold invariant under the restriction of the phase flow to the open set W . Condition (iii) means that the closed 2-form induced on M by the symplectic structure on P^{2n} is almost everywhere non-degenerate. Thus the flow on M is almost everywhere Hamiltonian with $n - k$ degrees of freedom. It inherits the property of complete integrability. Then almost all its integral manifolds consist of $(n - k)$ -dimensional tori and therefore lie in the critical set of the momentum map. Since M is closed in W and the critical set is closed in P^{2n} , we conclude that M totally consists of the critical points of the momentum map. \square

Remark 2. In our case $n = 3$ and the above lemma is applied in the situations when $k = 1$ or $k = 2$. The critical set and the bifurcation diagram of the map (3.16) in the case $\lambda = 0$ are known. The critical set is described by one system of the type (3.17) with $k = 2$ and three systems of the type (3.17) with $k = 1$. The complete presentation of these results and the list of publications are given in [12], [17]. Except for the partial integrable case of Bogoyavlensky [3] (case $K = 0$), all of the critical subsystems have been either explicitly integrated or reduced to separated systems of equations [13], [14], [16].

Introduce the change of variables [10] based on the change given by S. Kowalevski and on the Lax representation [2] ($i^2 = -1$)

$$\begin{aligned} x_1 &= (\alpha_1 - \beta_2) + i(\alpha_2 + \beta_1), & x_2 &= (\alpha_1 - \beta_2) - i(\alpha_2 + \beta_1), \\ y_1 &= (\alpha_1 + \beta_2) + i(\alpha_2 - \beta_1), & y_2 &= (\alpha_1 + \beta_2) - i(\alpha_2 - \beta_1), \\ z_1 &= \alpha_3 + i\beta_3, & z_2 &= \alpha_3 - i\beta_3, \\ w_1 &= \omega_1 + i\omega_2, & w_2 &= \omega_1 - i\omega_2, & w_3 &= \omega_3. \end{aligned} \quad (3.18)$$

Then Eqs. (3.15) yield

$$\begin{aligned}
2w'_1 &= -w_1(w_3 - \lambda) - z_1, & 2w'_2 &= w_2(w_3 - \lambda) + z_2, & 2w'_3 &= y_2 - y_1, \\
x'_1 &= -x_1w_3 + z_1w_1, & x'_2 &= x_2w_3 - z_2w_2, \\
y'_1 &= -y_1w_3 + z_2w_1, & y'_2 &= y_2w_3 - z_1w_2, \\
2z'_1 &= x_1w_2 - y_2w_1, & 2z'_2 &= -x_2w_1 + y_1w_2.
\end{aligned} \tag{3.19}$$

Here prime stands for $d/d(it)$.

Consider (3.18) as the map $\mathbb{R}^9 \rightarrow \mathbb{C}^9$ and denote its image by V^9 . Eqs. (2.9) of the phase space P^6 in V^9 take the form

$$z_1^2 + x_1y_2 = r^2, \quad z_2^2 + x_2y_1 = r^2, \tag{3.20}$$

$$x_1x_2 + y_1y_2 + 2z_1z_2 = 2p^2. \tag{3.21}$$

Here we introduce the positive constants

$$p = \sqrt{a^2 + b^2}, \quad r = \sqrt{a^2 - b^2}.$$

Using (3.20) and (3.21), express the first integrals in new coordinates,

$$\begin{aligned}
H &= w_1w_2 + \frac{1}{2}w_3^2 - \frac{1}{2}(y_1 + y_2), \\
K &= (w_1^2 + x_1)(w_2^2 + x_2) + 2\lambda(w_1w_2w_3 + z_2w_1 + z_1w_2) - 2\lambda^2w_1w_2, \\
G &= \frac{1}{4}(p^2 - x_1x_2)w_3^2 + \frac{1}{2}(x_2z_1w_1 + x_1z_2w_2)w_3 + \\
&\quad + \frac{1}{4}(x_2w_1 + y_1w_2)(y_2w_1 + x_1w_2) - \frac{1}{4}p^2(y_1 + y_2) + \\
&\quad + \frac{1}{4}r^2(x_1 + x_2) + \frac{1}{2}\lambda(z_1z_2w_3 + y_2z_2w_1 + y_1z_1w_2) + \\
&\quad + \frac{1}{4}\lambda^2(p^2 - y_1y_2).
\end{aligned} \tag{3.22}$$

Let f be an arbitrary function on V^9 . For brevity, the term "critical point of f " will always mean a critical point of the restriction of f to P^6 . Similarly, df means the restriction of the differential of f to the set of vectors tangent to P^6 . While calculating critical points of various functions, it is convenient to avoid introducing Lagrange's multipliers for the restrictions (3.20) and (3.21).

Lemma 6. *Critical points of a function f on V^9 , in the above sense, are defined by the system of equations*

$$X_i f = 0 \quad (i = 1, \dots, 6), \tag{3.23}$$

where

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial w_1}, \quad X_2 = \frac{\partial}{\partial w_2}, \quad X_3 = \frac{\partial}{\partial w_3}, \\
X_4 &= z_2 \frac{\partial}{\partial x_2} + z_1 \frac{\partial}{\partial y_2} - \frac{1}{2}x_1 \frac{\partial}{\partial z_1} - \frac{1}{2}y_1 \frac{\partial}{\partial z_2}, \\
X_5 &= z_1 \frac{\partial}{\partial x_1} + z_2 \frac{\partial}{\partial y_1} - \frac{1}{2}y_2 \frac{\partial}{\partial z_1} - \frac{1}{2}x_2 \frac{\partial}{\partial z_2}, \\
X_6 &= x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial y_2}.
\end{aligned}$$

Indeed, six vector fields X_i are tangent to P^6 and linearly independent at any point of P^6 .

The following two propositions define the strata \mathfrak{C}_0 and \mathfrak{C}_1 of the critical set.

Proposition 1. *The set \mathfrak{C}_0 consists exactly of the four equilibria existing in this problem.*

Proof. The condition of zero rank of the momentum map at a point $\zeta \in P^6$ supposes, in particular, that $dH = 0$. Then ζ is the point of equilibrium and it follows from Lemma 3 that ζ is one of the points (2.11). Using the complex variables we have

$$\begin{aligned} w_1 = w_2 = w_3 = 0, \quad z_1 = z_2 = 0, \\ x_1 = x_2 = \varepsilon_1 a - \varepsilon_2 b, \quad y_1 = y_2 = \varepsilon_1 a + \varepsilon_2 b \quad (\varepsilon_1 = \pm 1, \quad \varepsilon_2 = \pm 1). \end{aligned}$$

Use Eqs. (3.23) with $f = K$ and $f = G$ to obtain that $dK(\zeta) = 0$ and $dG(\zeta) = 0$. Therefore, $\text{rank } J(\zeta) = 0$. \square

Note that in classical problems of the rigid body dynamics with an axially symmetric force field, the rank of the momentum map is everywhere not less than 1 due to the regularity of the cyclic integral. In our case, all equilibria are *non-degenerate* (in the Morse sense) critical points of the Hamilton function (see [15]). Therefore, these points are critical for any first integral of the system.

It is essential that in the sequel $\lambda \neq 0$.

Proposition 2. *The set \mathfrak{C}_1 is completely defined by the condition*

$$\text{rank}\{dK, dH\} = 1$$

and consists of the points of the following periodic trajectories:

- 1) *pendulum motions (2.14);*
- 2) *motions defined by the equations*

$$w_1 = q(w)\sqrt{w}, \quad w_2 = \frac{\sqrt{w}}{q(w)}, \quad w_3 = \frac{\lambda}{\sigma}w, \quad (3.24)$$

$$\begin{aligned} x_1 &= \frac{1}{\sigma u} [r^2 \lambda^2 \sigma^2 - (\lambda^2 + \sigma) u q^2(w) w], \\ x_2 &= \frac{1}{\sigma u} [r^2 \lambda^2 \sigma^2 - (\lambda^2 + \sigma) u \frac{w}{q^2(w)}], \\ y_1 &= \sigma \left(1 + \frac{\sigma}{\lambda^2} - \frac{r^4 \lambda^2 \sigma}{u^2}\right) + \frac{r^2 \lambda^2}{u} q^2(w) w, \\ y_2 &= \sigma \left(1 + \frac{\sigma}{\lambda^2} - \frac{r^4 \lambda^2 \sigma}{u^2}\right) + \frac{r^2 \lambda^2}{u} \frac{w}{q^2(w)}, \\ z_1 &= -\frac{r^2 \lambda \sigma}{u} \frac{\sqrt{w}}{q(w)} + \frac{\lambda^2 + \sigma}{\lambda} q(w) \sqrt{w}, \\ z_2 &= -\frac{r^2 \lambda \sigma}{u} q(w) \sqrt{w} + \frac{\lambda^2 + \sigma}{\lambda} \frac{\sqrt{w}}{q(w)}. \end{aligned} \quad (3.25)$$

Here $q(w)$ is the root of the equation $q^4 - 2Q(w)q^2 + 1 = 0$, where

$$Q(w) = \frac{\sigma u^3 + (\lambda^2 + \sigma)[\lambda^2 w^2 + \sigma^2(2w - \sigma)]u^2 + r^4 \lambda^4 \sigma^4}{2r^2 \lambda^2 \sigma^2 (\lambda^2 + \sigma) u w}; \quad (3.26)$$

σ, u are constants satisfying the equation

$$\begin{aligned} \lambda^2(\lambda^2 + \sigma)^2 u^5 + (\lambda^2 + \sigma)[2p^2 \lambda^4 - (\lambda^2 + \sigma)^3 \sigma] \sigma u^4 + \\ + r^4 \lambda^6 \sigma^2 u^3 + 2r^4 \lambda^4 \sigma^4 (\lambda^2 + \sigma)^2 u^2 - r^8 \lambda^8 \sigma^6 = 0. \end{aligned} \quad (3.27)$$

The evolution $w(t)$ is defined by the equation

$$\left(\frac{dw}{dt}\right)^2 = -\frac{\lambda^2}{4\sigma^2} P_+(w) P_-(w), \quad (3.28)$$

where

$$P_{\pm}(w) = w^2 + 2\sigma^2 \frac{u \pm r^2 \lambda^2}{\lambda^2 u} w + \frac{\sigma[u^3 - (\lambda^2 + \sigma)\sigma^2 u^2 + r^4 \lambda^4 \sigma^3]}{(\lambda^2 + \sigma)\lambda^2 u^2}. \quad (3.29)$$

Proof. It follows from above that $dH \neq 0$ at the points of \mathfrak{C}_1 . Then to investigate the dependence of the functions K and H it is sufficient to introduce the function with one Lagrange's multiplier σ . Write Eqs. (3.23) with $f = K - 2\sigma H$,

$$\begin{aligned} (w_1^2 + x_1)w_2 + \lambda[z_1 + w_1(w_3 - \lambda)] - \sigma w_1 &= 0, \\ (w_2^2 + x_2)w_1 + \lambda[z_2 + w_2(w_3 - \lambda)] - \sigma w_2 &= 0, \end{aligned} \quad (3.30)$$

$$\lambda w_1 w_2 - \sigma w_3 = 0, \quad (3.31)$$

$$\begin{aligned} (w_1^2 + x_1)z_2 - \lambda(w_2 x_1 + w_1 y_1) + \sigma z_1 &= 0, \\ (w_2^2 + x_2)z_1 - \lambda(w_1 x_2 + w_2 y_2) + \sigma z_2 &= 0, \end{aligned} \quad (3.32)$$

$$x_1 w_2^2 - x_2 w_1^2 + \sigma(y_1 - y_2) = 0. \quad (3.33)$$

First consider the critical points of the function K . For this purpose we must put $\sigma = 0$. Eq. (3.31) gives $w_1 = w_2 = 0$. Then Eqs. (3.30) imply $z_1 = z_2 = 0$. Eqs. (3.32) and (3.33) become identities. The same values satisfy Eqs. (3.23) if we take $f = 4G + (x_1 x_2 - y_1 y_2)H$. Therefore, $dK = 0$ and $4dG + (x_1 x_2 - y_1 y_2)dH = 0$. Since $dH \neq 0$, it means that $\text{rank } J = 1$. The initial variables on the corresponding trajectories are $\omega_1 = \omega_2 \equiv 0$, $\alpha_3 = \beta_3 \equiv 0$. Substitute these values to Eqs. (3.15) to obtain the solutions (2.14).

Let $\sigma \neq 0$. The equilibria of the system are already excluded. Then it follows from (3.31) that $w_1 w_2 \neq 0$. Satisfying (3.31), introduce new variables w, q as shown in (3.24). Four equations (3.30), (3.32) form the linear system in y_1, y_2, z_1, z_2 , from which we obtain these variables as the functions of x_1, x_2, w, q identically satisfying (3.33). Denote

$$u = (w - \sigma)^2(\lambda^2 + \sigma) - \sigma x_1 x_2. \quad (3.34)$$

Then Eqs. (3.20) are easily solved for x_1, x_2 as the functions of w, q, u . As a result we obtain the expressions (3.25). Let

$$Q = \frac{1}{2}(q^2 + \frac{1}{q^2}).$$

Then the substitution of x_1, x_2 from (3.25) back to (3.34) gives (3.26). The last unused equation (3.21) provides the relation (3.27) between u and the constants λ, σ . It shows that u defined as (3.34) appears to be a constant.

Thus, all phase variables are expressed via one variable w , for which from (3.19) we find the differential equation (3.28). Note that due to (3.29) the solutions are elliptic functions of time.

To finish the proof, we need to show that at the points of the trajectories found we really have $\text{rank } J = 1$, i.e., the linear dependence of dK and dH implies the linear dependence of dG and dH . Indeed, Eqs. (3.23) with

$$f = 2G - (p^2 + \frac{\lambda^2 + \sigma}{\lambda^2 \sigma} u)H$$

are satisfied both by (2.14) and by (3.24), (3.25). Therefore, $\text{rank}\{dG, dH\} = 1$ and, consequently, $\text{rank}\{dK, dG, dH\} = 1$. \square

The following statement describes one of the critical subsystems in \mathfrak{C}_2 .

Proposition 3. *The system (3.19) has the four-dimensional invariant submanifold \mathfrak{D}_* defined by the equations*

$$U_1 = 0, \quad U_2 = 0, \quad (3.35)$$

where

$$\begin{aligned} U_1 &= \frac{y_2 w_1 + x_1 w_2 + z_1(w_3 + \lambda)}{w_1} - \frac{x_2 w_1 + y_1 w_2 + z_2(w_3 + \lambda)}{w_2}, \\ U_2 &= w_1 w_2 U_1'. \end{aligned} \quad (3.36)$$

The Poisson bracket $\{U_1, U_2\}$ is non-zero almost everywhere on this submanifold.

Proof. The derivative U_2' in virtue of (3.19) is proportional to U_1 , i.e., (3.35) implies $U_2' = 0$. Therefore the set (3.35) is invariant.

Consider the function

$$S = -\frac{1}{4} \left[\frac{y_2 w_1 + x_1 w_2 + z_1 (w_3 + \lambda)}{w_1} + \frac{x_2 w_1 + y_1 w_2 + z_2 (w_3 + \lambda)}{w_2} \right].$$

On \mathfrak{D}_* we obtain

$$S' = -\frac{w_1 z_2 + w_2 z_1 + w_1 w_2 (w_3 - \lambda)}{8w_1 w_2} U_1 \equiv 0.$$

Therefore, S is a partial integral of the induced system. Eliminate y_1, y_2 with the help of Eqs. (3.35) and present S in a more simple form

$$S = \frac{x_2 z_1 w_1 + x_1 z_2 w_2 + z_1 z_2 (w_3 + \lambda)}{2w_1 w_2 (w_3 - \lambda)}. \quad (3.37)$$

Now the Poisson bracket of U_1 and U_2 calculated under the rules defined by (2.3) is expressed in terms of the energy constant h and the constant s of the integral (3.37) in the following way

$$\{U_1, U_2\} = -\frac{4}{s} \left[3s^4 - 2s^3 \left(h - \frac{\lambda^2}{2} \right) + \frac{p^4 - r^4}{4} \right].$$

Obviously, the right part of it is a ratio of polynomials not identically zero on \mathfrak{D}_* . Therefore the set $\{U_1, U_2\} = 0$ has codimension 1 in \mathfrak{D}_* . In particular this set is of zero measure in \mathfrak{D}_* . \square

Remark 3. If $\lambda = 0$, then the manifold \mathfrak{D}_* turns into the phase space of the Hamiltonian system with two degrees of freedom studied in [14]. The geometrical characteristic of the motions in this system is the condition

$$\frac{\mathbf{M} \cdot \boldsymbol{\alpha}}{\mathbf{M} \cdot \mathbf{e}_1} = \frac{\mathbf{M} \cdot \boldsymbol{\beta}}{\mathbf{M} \cdot \mathbf{e}_2} = \text{const},$$

where $\mathbf{M} = \mathbf{I}\boldsymbol{\omega}$ is the angular momentum vector. The system (3.35), (3.36) is found from the same condition given that here $\mathbf{M} = \mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}$.

The following theorem completes the description of the critical set of the momentum map for the gyrostat.

Theorem 2. *The set of critical points of the momentum map (3.16) consists of the following subsets in P^6 :*

1) the set \mathfrak{L} defined by the system

$$w_1 = 0, \quad w_2 = 0, \quad z_1 = 0, \quad z_2 = 0; \quad (3.38)$$

2) the set \mathfrak{N} defined by the system

$$F_1 = 0, \quad F_2 = 0, \quad (3.39)$$

where

$$\begin{aligned} F_1 &= (w_1 w_2 + \lambda w_3)(w_2 x_1 + \lambda z_1) \lambda y_1 - \\ &\quad - w_2 (w_1^2 + x_1)(x_2 z_1 w_1 + x_1 z_2 w_2 - x_1 x_2 w_3 + 2z_1 z_2 \lambda) - \\ &\quad - x_2 (w_1 w_3 + z_1)(w_1 z_1 - x_1 w_3) \lambda + (x_1 w_3^2 - 2z_1 w_1 w_3 - z_1^2) z_2 \lambda^2, \\ F_2 &= (w_1 w_2 + \lambda w_3)(w_1 x_2 + \lambda z_2) \lambda y_2 - \\ &\quad - w_1 (w_2^2 + x_2)(x_2 z_1 w_1 + x_1 z_2 w_2 - x_1 x_2 w_3 + 2z_1 z_2 \lambda) - \\ &\quad - x_1 (w_2 w_3 + z_2)(w_2 z_2 - x_2 w_3) \lambda + (x_2 w_3^2 - 2z_2 w_2 w_3 - z_2^2) z_1 \lambda^2; \end{aligned}$$

3) the set \mathfrak{D} defined by the system

$$R_1 = 0, \quad R_2 = 0, \quad (3.40)$$

where

$$\begin{aligned} R_1 &= [y_1 w_2 + x_2 w_1 + z_2(w_3 + \lambda)]w_1(w_3 - \lambda) + \\ &\quad + x_2 z_1 w_1 + x_1 z_2 w_2 + z_1 z_2(w_3 + \lambda), \\ R_2 &= [y_2 w_1 + x_1 w_2 + z_1(w_3 + \lambda)]w_2(w_3 - \lambda) + \\ &\quad + x_2 z_1 w_1 + x_1 z_2 w_2 + z_1 z_2(w_3 + \lambda). \end{aligned}$$

Proof. We need to prove that

$$\mathfrak{L} \cup \mathfrak{N} \cup \mathfrak{D} = \mathfrak{C}. \quad (3.41)$$

It follows from Propositions 1 and 2 that $\mathfrak{L} \subset \mathfrak{C}_0 \cup \mathfrak{C}_1$. Indeed on \mathfrak{L} we have $dK \equiv 0$, $dG \equiv \pm ab dH$. Note also that the system of relations (3.38) satisfies the conditions of Lemma 5 with $n = 3, k = 2$. Therefore, \mathfrak{L} is a smooth two-dimensional manifold with the induced Hamiltonian system with one degree of freedom.

According to Proposition 3 we have $\mathfrak{D}_* \subset \mathfrak{C}_2$. Hence, $\mathfrak{D} = \mathfrak{D}_* \cup \mathfrak{L} \subset \mathfrak{C}$.

Consider the set \mathfrak{N} defined by Eqs. (3.39). First, investigate the cases when these equations cannot be solved with respect to y_1, y_2 . Suppose that

$$w_1 w_2 + \lambda w_3 \equiv 0. \quad (3.42)$$

Then, after several differentiations in virtue of (3.19), we come to Eqs. (3.30) – (3.33) with $\sigma = -\lambda^2$. The corresponding points belong to \mathfrak{C}_1 . Let

$$(w_2 x_1 + \lambda z_1)(w_1 x_2 + \lambda z_2) \equiv 0. \quad (3.43)$$

Then the same procedure leads to the system of equations having the only solutions of the form (3.38), i.e., to the set \mathfrak{L} . Denote $\mathfrak{N}_* = \mathfrak{N} \setminus (\mathfrak{C}_0 \cup \mathfrak{C}_1)$. On this set from (3.39) we obtain

$$\begin{aligned} y_1 &= \frac{1}{(w_1 w_2 + \lambda w_3)(w_2 x_1 + \lambda z_1)\lambda} [w_2(w_1^2 + x_1)(x_2 z_1 w_1 + x_1 z_2 w_2 - \\ &\quad - x_1 x_2 w_3 + 2z_1 z_2 \lambda) + x_2(w_1 w_3 + z_1)(w_1 z_1 - x_1 w_3)\lambda - \\ &\quad - (x_1 w_3^2 - 2z_1 w_1 w_3 - z_1^2)z_2 \lambda^2], \\ y_2 &= \frac{1}{(w_1 w_2 + \lambda w_3)(w_1 x_2 + \lambda z_2)\lambda} [w_1(w_2^2 + x_2)(x_2 z_1 w_1 + x_1 z_2 w_2 - \\ &\quad - x_1 x_2 w_3 + 2z_1 z_2 \lambda) + x_1(w_2 w_3 + z_2)(w_2 z_2 - x_2 w_3)\lambda - \\ &\quad - (x_2 w_3^2 - 2z_2 w_2 w_3 - z_2^2)z_1 \lambda^2]. \end{aligned} \quad (3.44)$$

The derivatives of F_1 and F_2 in virtue of (3.19) vanish identically after the substitution of the expressions (3.44). This fact proves that \mathfrak{N}_* is an invariant set. The Poisson bracket $\{F_1, F_2\}$ with (3.44) takes the form

$$\{F_1, F_2\} = \sqrt{2} \lambda (w_1 w_2 + \lambda w_3)^{3/2} \sqrt{(w_2 x_1 + \lambda z_1)(w_1 x_2 + \lambda z_2)} C. \quad (3.45)$$

Here

$$C = \frac{1}{s} (8s^3 \lambda^2 - r^4) \sqrt{2s^2 - (2h + \lambda^2)s + p^2} \quad (3.46)$$

depends on the energy constant h and the constant s of the partial integral

$$S = \frac{x_1 x_2 w_3 - x_2 z_1 w_1 - x_1 z_2 w_2 - \lambda z_1 z_2}{2\lambda(w_1 w_2 + \lambda w_3)}. \quad (3.47)$$

This integral is similar to (3.37) and independent of H almost everywhere on \mathfrak{N}_* .

The cases when the multipliers (3.42) or (3.43) in (3.45) turn to zero are already studied above. Zeros of the function (3.46) have codimension 1. Hence, (3.45) is non-zero almost everywhere on \mathfrak{N}_* . It follows from Lemma 5 that $\mathfrak{N}_* \subset \mathfrak{C}_2$. Thus, $\mathfrak{L} \cup \mathfrak{N} \cup \mathfrak{D} \subset \mathfrak{C}$. To prove the equality (3.41), we must show that $\mathfrak{C} \subset \mathfrak{L} \cup \mathfrak{N} \cup \mathfrak{D}$.

The points of the set \mathfrak{C}_0 described in Proposition 1 satisfy (3.38). According to Proposition 2 the set \mathfrak{C}_1 can be represented as $\mathfrak{C}_{11} \cup \mathfrak{C}_{12}$, where \mathfrak{C}_{11} consists of the trajectories (2.14) and \mathfrak{C}_{12} is defined by the system (3.24)–(3.27). On the trajectories (2.14) we have (3.38). It is easily checked that the points given by Eqs. (3.24)–(3.26) under the condition (3.27) satisfy both systems (3.39) and (3.40). Therefore, $\mathfrak{C}_0 \cup \mathfrak{C}_{11} \subset \mathfrak{L}$ and $\mathfrak{C}_{12} \subset \mathfrak{N} \cap \mathfrak{D}$.

Consider now the set \mathfrak{C}_2 . To investigate the dependence of G, H, K , introduce the function with Lagrange's multipliers. It follows from Propositions 1 and 2 that on \mathfrak{C}_2 the differentials dK and dH are linearly independent. Then the multiplier at the function G is always non-zero and can be chosen equal to any non-zero constant. It is convenient to take the function $2G + SK + (T - p^2)H$, where S and T are Lagrange's undefined multipliers. The condition

$$2dG + S dK + (T - p^2) dH = 0 \quad (3.48)$$

is preserved by the phase flow. Applying the Lie derivative we obtain

$$\dot{S} dK + \dot{T} dH = 0.$$

Since $\text{rank}\{dG, dK, dH\} = 2$, this linear combination of the differentials is proportional to the left part of (3.48). It means that, at the points of \mathfrak{C}_2 ,

$$\dot{S} \equiv 0, \quad \dot{T} \equiv 0.$$

Thus, the functions S and T are the partial integrals on the submanifold \mathfrak{C}_2 . According to Lemma 6 rewrite Eq. (3.48) as the system

$$\begin{aligned} x_2(y_2 + 2S)w_1 + 2S(w_1w_2 + \lambda w_3)w_2 + \\ + (T - z_1z_2 - 2S\lambda^2)w_2 + x_2z_1w_3 + (y_2 + 2S)z_2\lambda = 0, \\ x_1(y_1 + 2S)w_2 + 2S(w_1w_2 + \lambda w_3)w_1 + \\ + (T - z_1z_2 - 2S\lambda^2)w_1 + x_1z_2w_3 + (y_1 + 2S)z_1\lambda = 0, \end{aligned} \quad (3.49)$$

$$\begin{aligned} (T - x_1x_2)w_3 + x_2z_1w_1 + x_1z_2w_2 + (2Sw_1w_2 + z_1z_2)\lambda = 0, \\ Tz_1 + x_1z_2w_3^2 + [(x_1x_2 - 2z_1z_2)w_1 + (y_1z_1 + x_1z_2)\lambda + x_1y_1w_2]w_3 - \\ - (y_1z_1 + x_1z_2)w_1w_2 + x_1(y_1 + 2S)w_2\lambda - \\ - [x_2z_1 + (y_2 + 2S)z_2]w_1^2 + [(y_2 + 2S)y_1 - 2z_1z_2]w_1\lambda + y_1z_1\lambda^2 - \\ - [(y_2 + 2S)x_1 + z_1^2]z_2 = 0, \\ Tz_2 + x_2z_1w_3^2 + [(x_1x_2 - 2z_1z_2)w_2 + (y_2z_2 + x_2z_1)\lambda + x_2y_2w_1]w_3 - \\ - (y_2z_2 + x_2z_1)w_1w_2 + x_2(y_2 + 2S)w_1\lambda - \\ - [x_1z_2 + (y_1 + 2S)z_1]w_2^2 + \\ + [(y_1 + 2S)y_2 - 2z_1z_2]w_2\lambda + y_2z_2\lambda^2 - \\ - [(y_1 + 2S)x_2 + z_2^2]z_1 = 0, \\ (T - x_1x_2)(y_1 - y_2) + 2(y_2 + S)x_2w_1^2 - 2(y_1 + S)x_1w_2^2 + \\ + 2(x_2z_1w_1 - x_1z_2w_2)w_3 + x_2z_1^2 - x_1z_2^2 + \\ + 2(y_2z_2w_1 - y_1z_1w_2)\lambda = 0. \end{aligned} \quad (3.50)$$

It follows from Proposition 3 that this system is valid at the points of the set \mathfrak{D}_* . To find all other cases suppose

$$U_1U_2 \neq 0 \quad (3.51)$$

and express y_1, y_2 from (3.36):

$$\begin{aligned} y_1 = \frac{1}{2w_1w_2(w_3 - \lambda)} \{2U_2 - [w_1w_2(w_3 - \lambda) + w_2z_1 - w_1z_2]U_1 - \\ - 2[w_1z_2(w_3 - \lambda)^2 + (x_2w_1^2 + z_1z_2 + 2\lambda w_1z_2)(w_3 - \lambda) + \\ + x_2z_1w_1 + x_1z_2w_2 + 2\lambda z_1z_2]\}, \\ y_2 = \frac{1}{2w_1w_2(w_3 - \lambda)} \{2U_2 + [w_1w_2(w_3 - \lambda) + w_1z_2 - w_2z_1]U_1 - \\ - 2[w_2z_1(w_3 - \lambda)^2 + (x_1w_2^2 + z_1z_2 + 2\lambda w_2z_1)(w_3 - \lambda) + \\ + x_2z_1w_1 + x_1z_2w_2 + 2\lambda z_1z_2]\}. \end{aligned} \quad (3.52)$$

The determinant of the system (3.49) with respect to $T, 2S$ is equal to

$$\Delta = x_1 w_2^2 - x_2 w_1^2 - (z_2 w_1 - z_1 w_2) \lambda.$$

If we suppose that $\Delta \equiv 0$ on some time interval, then the sequence of the derivatives of this identity in virtue of (3.19) leads to (3.38), i.e., to the points of $\mathfrak{C}_0 \cup \mathfrak{C}_1$. Consider then

$$\Delta \neq 0 \tag{3.53}$$

and find from Eqs. (3.49)

$$\begin{aligned} S &= \frac{1}{\Delta} [x_2 y_2 w_1^2 - x_1 y_1 w_2^2 + (x_2 z_1 w_1 - x_1 z_2 w_2) w_3 + \\ &\quad + (y_2 z_2 w_1 - y_1 z_1 w_2) \lambda], \\ T &= \frac{1}{\Delta} [A_1 B_1 - A_2 B_2]. \end{aligned} \tag{3.54}$$

Here

$$\begin{aligned} A_1 &= (x_1 w_2 + \lambda z_1) y_1 + (x_1 w_3 - z_1 w_1) z_2, \\ B_1 &= (w_2^2 + x_2) w_1 + \lambda w_2 (w_3 - \lambda) + \lambda z_2, \\ A_2 &= (x_2 w_1 + \lambda z_2) y_2 + (x_2 w_3 - z_2 w_2) z_1, \\ B_2 &= (w_1^2 + x_1) w_2 + \lambda w_1 (w_3 - \lambda) + \lambda z_1. \end{aligned}$$

Substitute (3.52) and (3.54) into (3.50) to obtain the system of four equations of the type $E_i = 0$ ($i = 1, \dots, 4$), where $E_i = a_{i2} U_1^2 + a_{i1} U_1 + a_{i0} U_2$ with some polynomials a_{ij} . For the proof of the theorem, there is no need to use all equations of this system. It is enough to consider, for example, the zero points of the resultant of two simplest functions E_1, E_4 with respect to U_2 . We obtain $4w_1 w_2 U_1 R \Delta = 0$, where

$$\begin{aligned} R &= \lambda w_1 w_2 (x_1 w_2 + \lambda z_1) (x_2 w_1 + \lambda z_2) U_1 - \{w_1 w_2 [x_2 z_1 w_1 + x_1 z_2 w_2 - \\ &\quad - x_1 x_2 (w_3 - \lambda) + 2\lambda z_1 z_2] + (z_1 z_2 w_3 + x_2 z_1 w_1 + x_1 z_2 w_2) \lambda^2 + z_1 z_2 \lambda^3\} \Delta. \end{aligned}$$

Due to (3.51), $U_1 \neq 0$. At the points of $\mathfrak{C} \setminus \mathfrak{L}$ the product $w_1 w_2$ is not identically zero. The set $\mathfrak{C} \setminus (\mathfrak{L} \cup \mathfrak{D})$ is, obviously, preserved by the phase flow. Therefore, (3.53) and (3.19) imply $R = 0, R' = 0$. These equations are linear in y_1, y_2 and yield the expressions (3.44) satisfying Eqs. (3.39). Thus, $\mathfrak{C} \setminus (\mathfrak{L} \cup \mathfrak{D}) \subset \mathfrak{N}$. \square

Remark 4. The system (3.40) follows from Eqs. (3.35) and (3.36). In particular, $\mathfrak{D}_* = \mathfrak{D} \setminus \mathfrak{L}$. Then from Proposition 3 and Lemma 5 we obtain that this manifold lies completely in \mathfrak{C}_2 .

Remark 5. The integral S given by (3.54), when restricted to the set \mathfrak{N} has the form (3.47). Indeed, it is enough to substitute (3.44) into the first formula (3.54) to obtain (3.47). On the set \mathfrak{D} the same function S in the substitution of (3.35) and (3.52) takes the form (3.37). Therefore, the use of the same notation in (3.37) and (3.47) is correct. The expressions for T can also be simplified. On the set \mathfrak{N} we have $T = 2\lambda^2 S$, i.e., this function does not give rise to a new partial integral independent of S . On the contrary, at the points of the set \mathfrak{D} we have

$$T = x_1 x_2 + z_1 z_2 - 2w_1 w_2 S.$$

In the case $\lambda = 0$ the same expression with the corresponding function S is the partial integral independent of S . The equations of the integral manifold defined by the pair S, T lead to the separation of variables on \mathfrak{D} [14].

4 The bifurcation diagram

The Lax representation for the considered problem found in [2] can be written in the form

$$L' = LM - ML, \tag{4.55}$$

where

$$L = \begin{pmatrix} 2\lambda & \frac{x_2}{\varkappa} & -2w_1 & \frac{z_2}{\varkappa} \\ -\frac{x_1}{\varkappa} & -2\lambda & -\frac{z_1}{\varkappa} & 2w_1 \\ -2w_1 & \frac{z_2}{\varkappa} & -2w_3 & -\frac{y_1}{\varkappa} - 4\varkappa \\ -\frac{z_1}{\varkappa} & 2w_2 & \frac{y_2}{\varkappa} + 4\varkappa & 2w_3 \end{pmatrix}, \quad M = \begin{pmatrix} -\frac{w_3}{2} & 0 & \frac{w_2}{2} & 0 \\ 0 & \frac{w_3}{2} & 0 & -\frac{w_1}{2} \\ \frac{w_1}{2} & 0 & \frac{w_2}{2} & \varkappa \\ 0 & -\frac{w_2}{2} & -\varkappa & -\frac{w_3}{2} \end{pmatrix}.$$

Here \varkappa stands for the spectral parameter, the derivative in (4.55) is calculated in virtue of the system (3.19). The equation for the eigenvalues μ of the matrix L defines the algebraic curve associated with this representation [8]. Let $s = 2\varkappa^2$ and let h, k, g be the arbitrary constants of the integrals (3.22). The equation of the algebraic curve takes the form

$$\mu^4 - 4\mu^2 \left[\frac{p^2}{s} - (2h + \lambda^2) + 2s \right] + 4 \left[\frac{r^4}{s^2} + \frac{2}{s} (4g - 2p^2h - p^2\lambda^2) + 4(k + 2\lambda^2h) - 8\lambda^2s \right] = 0. \quad (4.56)$$

It is natural to suppose that the bifurcation diagram of the momentum map (3.16) is included in the set of values (g, k, h) such that the curve (4.56) either have singular points or is reducible, i.e., the left part of Eq. (4.56) splits into the product of some rational non-trivial expressions. In this way we can guess the result of the following statement. Nevertheless, to obtain the complete proof of it, we must fulfil the calculations on the above found critical manifolds.

Theorem 3. *The bifurcation diagram of the momentum map $G \times K \times H$ is included in the union of the following (intersecting) subsets of $\mathbb{R}^3(g, k, h)$:*

1) the pair of straight lines

$$\Gamma_+ : \begin{cases} k = (a+b)^2, \\ g = -ab(h - \frac{\lambda^2}{2}); \end{cases} \quad \Gamma_- : \begin{cases} k = (a-b)^2, \\ g = ab(h - \frac{\lambda^2}{2}); \end{cases} \quad (4.57)$$

2) the surface

$$\Gamma_1 : \begin{cases} k = 4\lambda^2s - 2\lambda^2h + \frac{r^4}{4s^2}, \\ g = -\lambda^2s^2 + \frac{1}{2}p^2(h + \frac{\lambda^2}{2}) - \frac{r^4}{4s}, \end{cases} \quad s \in \mathbb{R} \setminus \{0\}; \quad (4.58)$$

3) the surface

$$\Gamma_2 : \begin{cases} k = 3s^2 - 4(h - \frac{\lambda^2}{2})s + p^2 + (h - \frac{\lambda^2}{2})^2 - \frac{p^4 - r^4}{4s^2}, \\ g = -s^3 + (h - \frac{\lambda^2}{2})s^2 + \frac{p^4 - r^4}{4s}, \end{cases} \quad s \in \mathbb{R} \setminus \{0\}. \quad (4.59)$$

Proof. Let $\zeta \in \mathfrak{L}$. Substitution of the values $z_1 = z_2 = 0$ into (3.20) and (3.21) yields $x_1x_2 = (a \pm b)^2$, $y_1y_2 = (a \mp b)^2$. Then from (3.22), (3.38) we obtain the equations defining the lines (4.57).

Let $\zeta \in \mathfrak{N} \setminus \mathfrak{L}$. Take the constant of the partial integral (3.47) for the parameter s in (4.58), substitute the expressions (3.22) for the corresponding constants, and fulfil the change (3.44). Then both Eqs. (4.58) become the identities. Therefore, $J(\mathfrak{N} \setminus \mathfrak{L}) \subset \Gamma_1$.

The inclusion $J(\mathfrak{D} \setminus \mathfrak{L}) \subset \Gamma_2$ is proved in a similar way. We take the constant of the partial integral (3.37) for the parameter s in (4.59) and fulfil the substitution (3.52) with $U_1 = U_2 = 0$. \square

Remark 6. Note that the shift of the energy level $\tilde{h} = h - \lambda^2/2$ makes the equations of the lines Γ_{\pm} and the surface Γ_2 independent of λ . Thereby obtained equations are identical with the corresponding equations of the case $\lambda = 0$ [12]. The surface Γ_1 is obtained as a perturbation (with respect to λ) of two tangent to each other sheets of the bifurcation diagram of the case $\lambda = 0$, i.e., the plane $k = 0$ and the slanted parabolic cylinder $(p^2h - 2g)^2 - r^4k = 0$. Thus, it is easy to view the evolution of the Appelrot classes [20] of the S.Kowalevski case in the process of two-way generalizations—adding the second force field and, afterwards, the non-zero gyrostatic momentum.

The equations given in Theorem 3 are in the following sense convenient. Let us fix the energy constant h . Then we obtain the parametric equations of a one-dimensional set in the plane (g, k) (with the finite number of singular points). This set is the bifurcation diagram Σ_h of the restriction of the pair of integrals G, K onto the iso-energetic surface $\{H = h\} \subset P^6$, which is always compact. In particular, all diagrams Σ_h lie in the restricted area of the (g, k) -plane and are easily drawn numerically. The analytical investigation of the types of the diagrams Σ_h with respect to the essential parameters $(b/a, \lambda/\sqrt{a}, h/a)$ is a necessary but technically complicated problem. Nevertheless, it must be solvable. Indeed, the set of double points and cusps of the curves $\Gamma_{1,2}$ in the (g, k) -plane is easily defined and investigated analytically. Moreover, the values of the first integrals on the periodic motions (3.24)–(3.27) define the points of transversal intersections $\Gamma_1 \cap \Gamma_2$. This fact, at least, guarantees that the numerical algorithm can be built for effective calculation of knots of one-dimensional cell complex Σ_h for any h . In turn, it should be possible to find all cases of bifurcations of the set of these knots with respect to the parameters defining the above set of periodic motions.

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